Quadratic map modulated by additive periodic forcing

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We study the effects on the quadratic map $x_{n+1}=\lambda-x_n^2$ of an additive periodic forcing term of period 2. The modulated system can possess two attractors which are noncomplementary. The system displays bistability and can exist in two different states of periodicity or chaos for a given λ. The two attractors coexist until, with increasing λ, the trajectory boundary of attractor I collides with the unstable fixed point created with attractor II and attractor I is destroyed. Thereafter, the system possesses only one attractor, except in periodic windows of the first attractor, where again the system has two attractors. For still larger λ, the trajectory boundary of attractor II also collides with the unstable fixed point that was created with it, resulting in a sudden enlargement of the chaotic regime. For even larger λ, in periodic windows, two attractors can again coexist. The range of λ for which the system is iteratively stable may not be continuous. We conclude by generalizing the broad features of period-2 forcing for the case of period-p forcing when the system can possess a maximum of p noncomplementary attractors.

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I. INTRODUCTION

The logistic map

$$y_{n+1}=ay_n(1-y_n) \quad (0\le y\le 1\ ,\ 0\le a\le 4)\ , \tag{1.1}$$

as representative of discrete one-dimensional maps with a quadratic maximum, has been studied very extensively from the point of view of deterministic chaos and the universality exhibited by such systems [1–3]. The logistic map modulated by multiplicative [4–8] or additive [8–11] forcing has been investigated both for random as well as periodic forcing. A feature of such systems when periodically forced is that they can possess more than one attractor. The aim of this paper is to explore systematically the behavior of quadratic maps modulated by periodic additive forcing. For this purpose we consider the symmetric quadratic map

$$x_{n+1}=\lambda-x_n^2\equiv F(x_n)\ , \tag{1.2}$$

which can be obtained by translation and rescaling from the logistic map (1.1). Such a system with additive periodic forcing can be described by the set of p equations:

$$x_{pn+j+1}=\lambda-x_{pn+j}^2+\epsilon\cos(2\pi j/p)\equiv f_{(p),j}(x_{pn+j})$$
$$(j=0,1,\ldots,p-1 \text{ and } n=0,1,2,\ldots)\ , \tag{1.3}$$

where p is the period and ϵ is the amplitude of the additive forcing. A direct product of the p maps $f_{(p),j}$ taken in sequence gives rise to a period-p map

$$f_{p,j}=f_{(p),j-1}\circ f_{(p),j-2}\circ\cdots\circ f_{(p),0}\circ f_{(p),p-1}\circ\cdots\circ f_{(p),j+1}\circ f_{(p),j}\ . \tag{1.4}$$

The cyclic permutation of the maps $f_{(p),j}$ in the direct product above define the p different period-p maps $f_{p,0},f_{p,1},\ldots,f_{p,p-1}$.

Additive period-1 forcing of the quadratic map is equivalent to a trivial redefinition of the parameter λ. We therefore begin our considerations with the study of the quadratic map modulated by additive period-2 forcing. In Sec. II we show that the two attractors born as a result of additive period-2 forcing are noncomplementary. In Sec. III we see that additive forcing causes a tangent bifurcation to occur in place of the first pitchfork bifurcation of the quadratic map. The system displays bistability and its two attractors can be in different states of periodicity or chaos for the same value of λ. In Sec. IV we describe the manner in which these attractors are destroyed as λ is increased. In Sec. V we study the effect of forcing on periodic windows of the quadratic map and in Sec. VI we conclude with some general remarks about the quadratic map modulated by an additive period-p forcing term.

II. EFFECTS OF ADDITIVE PERIOD-2 FORCING

Consider the quadratic map (1.2) driven by an additive period-2 forcing term and described by the equations

$$x_{2n+1}=\lambda-x_{2n}^2+\epsilon\equiv f_+(x_{2n})\ , \tag{2.1}$$

$$x_{2n+2}=\lambda-x_{2n+1}^2-\epsilon\equiv f_-(x_{2n+1})\ . \tag{2.2}$$

We can define two maps f_α and f_β obtained by successive alternate applications of the maps f_+ and f_-:

$$x_{2n+1}=f_+(x_{2n})=f_+(f_-(x_{2n-1}))\equiv f_\alpha(x_{2n-1})\ , \tag{2.3}$$

$$x_{2n+2}=f_-(x_{2n+1})=f_-(f_+(x_{2n}))\equiv f_\beta(x_{2n})\ . \tag{2.4}$$

Note that f_α and f_β, after suitable scaling, can be related to the quartic map studied by Chang, Wortis, and Wright [12].

Since $f_\alpha\equiv f_+\circ f_-$ and $f_\beta\equiv f_-\circ f_+$ and as f_- is a locally invertible map, we can write

$$f_+ \circ f_- = f_-^{-1} \circ (f_- \circ f_+) \circ f_- \;. \tag{2.5}$$

Therefore

$$f_\alpha = f_-^{-1} \circ f_\beta \circ f_- \tag{2.6}$$

and similarly

$$f_\beta = f_+^{-1} \circ f_\alpha \circ f_+ \;, \tag{2.7}$$

i.e., the maps f_α and f_β are topological conjugates [12–14]. If x_α^* is a fixed point of f_α then $f_-(x_\alpha^*)$ will be a fixed point of f_β and if x_β^* is a fixed point of f_β then $f_+(x_\beta^*)$ will be a fixed point of f_α. The state of periodicity of f_β will be identical to that of f_α for every λ and ϵ. The properties of the map f_β can therefore be inferred from the properties of the map f_α. We shall therefore choose to focus our attention primarily on the map $f_\alpha(x)$.

The quadratic map $F(x)$ can be viewed as the half-cycle map [15] of $F^{(2)}(x)$, which is the second iterate of $F(x)$. At values of λ for which $F(x)$ has an attractor of period $2n$, $F^{(2)}(x)$ has two complementary attractors of period n. Attractors are considered to be complementary if they are identical when seen from the standpoint of half-cycle maps.

We will now show that the two attractors of the map $f_\alpha(x)$ are not complementary. Consider a value of λ for which $f_\alpha(x)$ possesses two *different* stable fixed points $x_{\alpha 1}^*$ and $x_{\alpha 2}^*$. Two asymptotic sequences of x under iteration by the half-cycle maps f_+ and f_- are now possible:

$$x_{\alpha 1}^* \underset{f_-}{\rightarrow} x_{\beta 1}^* \underset{f_+}{\rightarrow} x_{\alpha 1}^* \underset{f_-}{\rightarrow} x_{\beta 1}^* \underset{f_+}{\rightarrow} \cdots \;, \tag{2.8}$$

$$x_{\alpha 2}^* \underset{f_-}{\rightarrow} x_{\beta 2}^* \underset{f_+}{\rightarrow} x_{\alpha 2}^* \underset{f_-}{\rightarrow} x_{\beta 2}^* \underset{f_+}{\rightarrow} \cdots \;. \tag{2.9}$$

If $x_{\alpha 1}^*$ is a fixed point of $f_\alpha(x)$ on one attractor then $x_{\beta 1}^*$ will be a fixed point of $f_\beta(x)$ on that attractor and these two fixed points will iterate to each other through the appropriate half-cycle maps f_+ or f_-. Similarly, for the other attractor, if $x_{\alpha 2}^*$ is a fixed point of $f_\alpha(x)$, $x_{\beta 2}^*$ will be a fixed point of $f_\beta(x)$.

The sequences (2.8) and (2.9) can be identical only if either (i)

$$x_{\alpha 1}^* = x_{\alpha 2}^* \quad \text{and} \quad x_{\beta 1}^* = x_{\beta 2}^*$$

or (ii)

$$x_{\alpha 1}^* = x_{\beta 2}^* \quad \text{and} \quad x_{\alpha 2}^* = x_{\beta 1}^* \;.$$

Since, by assumption, the two stable fixed points $x_{\alpha 1}^*$ and $x_{\alpha 2}^*$ are not identical, (i) cannot be true. In case (ii) the sequences (2.8) and (2.9) become

$$x_{\alpha 1}^* \underset{f_-}{\rightarrow} x_{\beta 1}^* \underset{f_+}{\rightarrow} x_{\alpha 1}^* \underset{f_-}{\rightarrow} x_{\beta 1}^* \underset{f_+}{\rightarrow} \cdots \;, \tag{2.10}$$

$$x_{\beta 1}^* \underset{f_-}{\rightarrow} x_{\alpha 1}^* \underset{f_+}{\rightarrow} x_{\beta 1}^* \underset{f_-}{\rightarrow} x_{\alpha 1}^* \underset{f_+}{\rightarrow} \cdots \;. \tag{2.11}$$

Thus $x_{\beta 1}^* = f_-(x_{\alpha 1}^*)$ from (2.10) and $x_{\beta 1}^* = f_+(x_{\alpha 1}^*)$ from (2.11), i.e.,

$$x_{\beta 1}^* = \lambda - (x_{\alpha 1}^*)^2 - \epsilon = \lambda - (x_{\alpha 1}^*)^2 + \epsilon \;,$$

which can be true only if $\epsilon = 0$. Therefore, for nonzero values of ϵ, the two sequences (2.8) and (2.9) cannot be identical. Consequently the two attractors (one with $x_{\alpha 1}^*$ and $x_{\beta 1}^*$ and the other with $x_{\alpha 2}^*$ and $x_{\beta 2}^*$ lying on their respective trajectories when seen from the standpoint of the half-cycle maps) are not complementary.

III. FIXED POINTS, BIFURCATIONS, AND BASINS OF ATTRACTION

The fixed points of the map $f_\alpha(x)$ are the zeros of $f_\alpha(x) - x$, which is a polynominal of the fourth degree in x. At λ_1 the first tangent bifurcation occurs and two complex-conjugate roots of the polynomial become real and coincident. Therefore using the method of highest common divisor and perturbation theory, one can show that $\lambda_1 \simeq -0.25(1-\epsilon^2)$ for small ϵ. As a result of this tangent bifurcation at λ_1, the system possesses two fixed points, one unstable x_a^* and the other stable x_b^*.

Let us denote by λ_2 the value of λ at which two new fixed points, one unstable x_c^* and the other stable x_d^*, appear as a result of a second tangent bifurcation. Using singular perturbation theory [16], it can be shown that $\lambda_2 \simeq 0.75 + (27\epsilon^2/4)^{1/3}$ for small ϵ. The period-2 forcing term causes the fixed points x_b^*, x_c^*, and x_d^*, whose counterparts for $F^{(2)}(x)$ were coincident at $\lambda = 0.75$, to be pushed apart. As shown in Sec. II, the two stable attractors x_b^* and x_d^* can not be complementary and therefore the forcing term acts like an asymmetric perturbation [15]. The separation between x_b^* and x_d^* at λ_2 increases with increasing ϵ. With increasing λ the fixed points x_b^* and x_d^* become unstable at different values of λ. But at whichever value of λ either becomes unstable; thereafter for increasing λ it undergoes a period-doubling cascade to chaos via pitchfork bifurcations with the same Feigenbaum constants as those of the standard quadratic map.

Since the two stable attractors x_b^* and x_d^* bifurcate at different values of λ, their periodicities for a given λ can be different, i.e., the system displays bistability. The difference in periodicities of the two attractors for a given λ increases with increasing ϵ. In fact, for sufficiently large ϵ, attractor I may already be in a chaotic state before the appearance of attractor II.

The basin boundaries of the two attractors keep changing with λ. Another effect of period-2 forcing is that the basin of attractor I (created at λ_1) is in general larger than the basin of attractor II (created at λ_2). Thus for random initial values of x and fixed $\lambda > \lambda_2$, the system usually converges to attractor I with higher probability. Such a phenomenon has been seen in nonequilibrium systems where an asymmetric interaction can select, with different probabilities, one of two possible states of equilibrium [17].

IV. CRISES OF THE ATTRACTORS

For $\lambda > \lambda_2$, the two attractors of the system coexist until they undergo crises [18] when their respective trajectory boundaries intersect the unstable fixed point x_c^* at different values of λ. This can be seen easily using box-construction techniques [12]. In Fig. 1 we plot $f_\alpha(x)$ as a

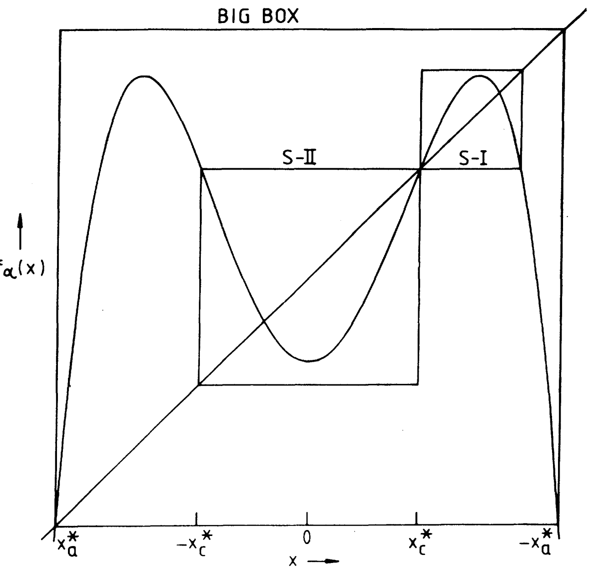

FIG. 1. The big box and the two small boxes of the map $f_\alpha(x)$ shown for $\epsilon=0.01$ and $\lambda=1.45$.

function of x for fixed $\lambda>\lambda_2$. One pair of opposite corners of the big box is located at (x_a^*, x_a^*) and $(-x_a^*, -x_a^*)$. If all the extrema of $f_\alpha(x)$ lie within the box, i.e., are (big) boxable, the system is iteratively stable for all x lying within the big box. In a similar way, as long as the right maximum and the central minimum of $f_\alpha(x)$ are both boxable within the small boxes S-I and S-II, which are separated by the unstable fixed point x_c^*, the two attractors continue to coexist.

Let us denote by λ_{c1} and λ_{c2}, respectively, the values of λ for which the trajectory boundaries of attractors I and II intersect the unstable fixed point x_c^*. Using perturbation theory one can show that $\lambda_{c1,c2}\simeq 1.5437\mp 1.5249\epsilon$ for small ϵ. At λ_{c1} the right maximum of $f_\alpha(x)$ just touches the top of small box S-I (see Fig. 1). For $\lambda>\lambda_{c1}$, the lower boundary of the trajectory of attractor I becomes less than x_c^*. Trajectories of attractor I can therefore escape from small box S-I and intrude into the basin of attractor II so that whenever attractor I is chaotic the system converges to attractor II for all initial x_0. In other words, at λ_{c1} a crisis occurs, resulting in the sudden destruction of the basin of attractor I. If attractor II is in a periodic state for $\lambda=\lambda_{c1}$, a transfer is seen from chaos (of attractor I) to periodicity (of attractor II) and this transition has been called a "transfer crisis" [10]. For λ slightly greater than λ_{c1}, an initial point within a region that was formerly the basin of attractor I will generate a chaotic transient until "by chance" it falls out of the former basin and is "caught" by the other attractor. If $\Psi(\tau)$ be the fraction of initial points, uniformly distributed in the former basin of attractor I, which has a chaotic transient of length τ, we find that

$$\Psi(\tau)\propto\exp(-\beta\tau) \tag{4.1}$$

i.e., the length of a chaotic transient is exponentially distributed [19]. Also, the average length of a chaotic transient decreases with increasing λ for $\lambda>\lambda_{c1}$.

As the value of λ is increased beyond λ_{c1}, at $\lambda=\lambda_{c2}$ the minimum of $f_\alpha(x)$ touches the small box S-II, i.e., the trajectory boundary of attractor II collides with the unstable fixed point x_c^* and this leads to a sudden enlargement of the chaotic regime. For $\lambda\geq\lambda_{c2}$, trajectories of

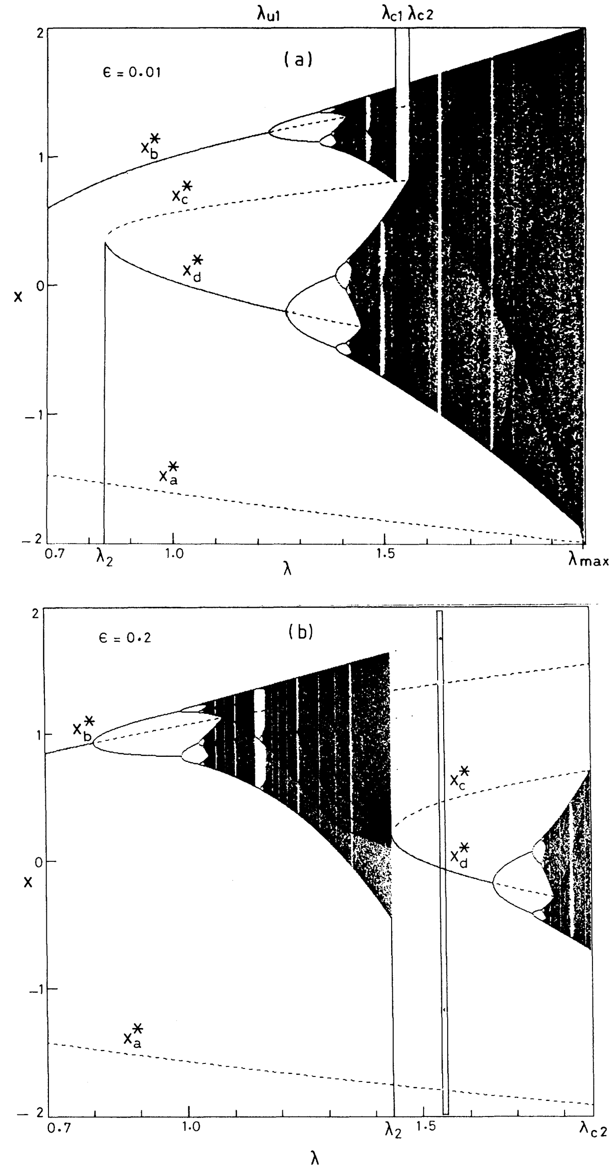

FIG. 2. The composite bifurcation diagram of $f_\alpha(x)$. (a) for $\epsilon=0.01$ shows the presence of two attractors and their crises when their trajectory boundaries meet the unstable fixed point x_c^*. (b) for $\epsilon=0.2$ shows the system converging to attractor II in the interval (λ_2,λ_{c2}) except within periodic windows of attractor I such as, for example, the region enclosed in the narrow rectangular box. On enlargement, one can see that within the box a period-2 window of attractor I coexists with a period-1 state of attractor II. In the interval $(\lambda_{c1},\lambda_{c2})$ normally only attractor II exists.

the system keep shuttling between the two regions which were formerly the basins of the two attractors. If we denote by $\langle t_{\rm I} \rangle$ and $\langle t_{\rm II} \rangle$ the average occupation times in the two regions, for λ slightly greater than λ_{c2}, it is seen that $\langle t_{\rm II} \rangle > \langle t_{\rm I} \rangle$ and the two become comparable for larger λ.

Let us denote by $\lambda_{\max}$ the value of λ at which the trajectory of attractor I, or the trajectory of the attractor which results after attractor II has undergone a crisis at λ_{c2}, collides with the unstable fixed point x_α^* (created at the first tangent bifurcation). Using perturbative methods one can show that $\lambda_{\max} \simeq 2 - 9\epsilon/5$ for small ϵ. In terms of the box construction procedure, $\lambda_{\max}$ is the smallest value of λ for which an extremum of $f_\alpha(x)$ just touches the big box. For small values of ϵ, the system is iteratively unstable for $\lambda > \lambda_{\max}$, i.e., its iterative domain [12] is an empty set.

In Fig. 2(a) we show the "composite" bifurcation diagram of the map $f_\alpha(x)$ for $\epsilon = 0.01$ and λ in the interval $(0.7, \lambda_{\max})$. By "composite," we mean that it is an overlap of bifurcation diagrams obtained by starting with different initial x_0 uniformly distributed over the iterative domain. Such an overlap enables us to see both attractors of the map, whenever they exist, in the same bifurcation diagram. It also suppresses the appearance of sudden jumps in the bifurcation diagram which would otherwise occur as a result of "basin crossing" [11]. The two attractors of $f_\alpha(x)$ and the crises occurring at λ_{c1} and λ_{c2} can be seen in the figure.

The interval $(\lambda_2, \lambda_{c1})$ over which the system has two

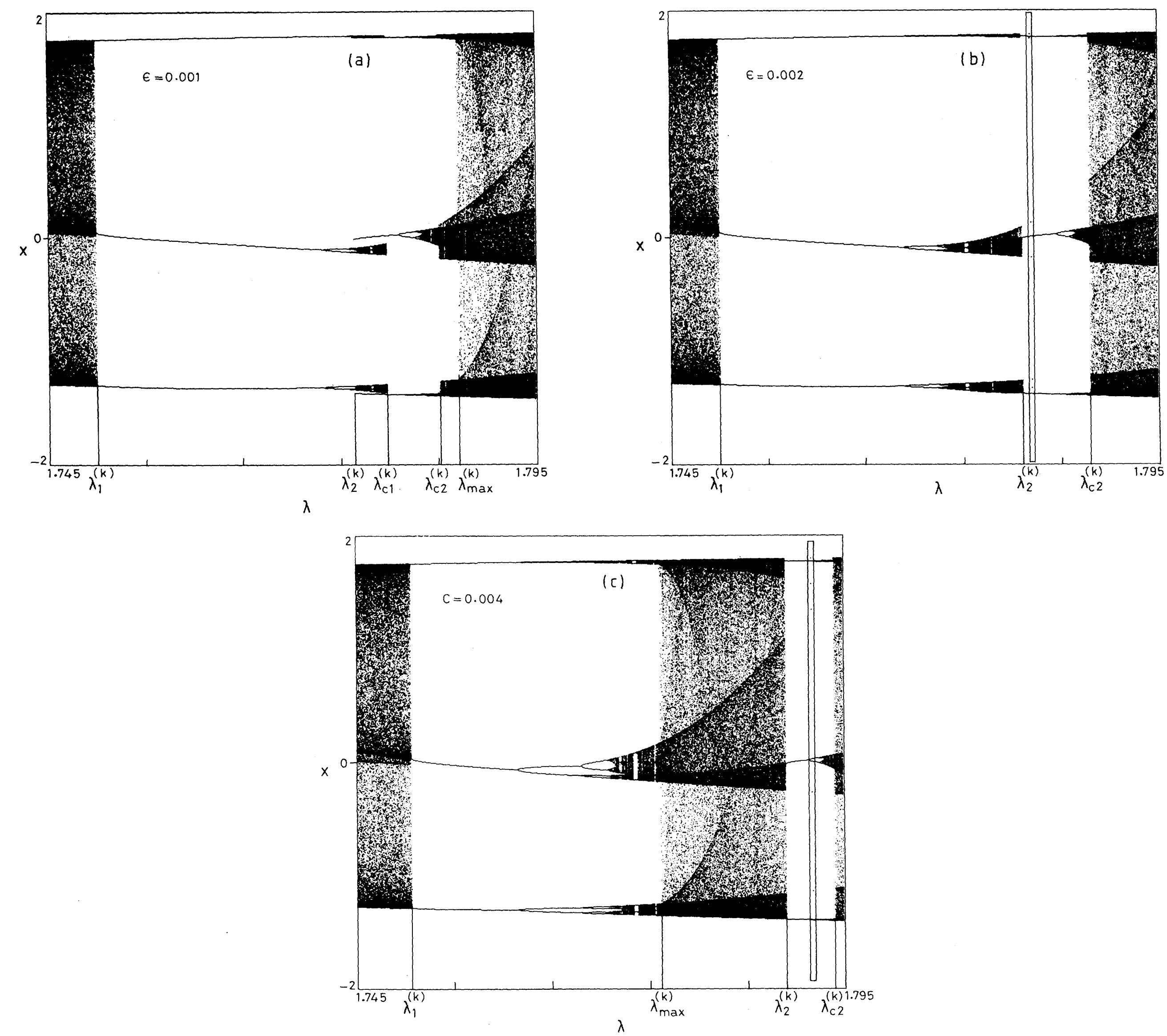

FIG. 3. The composite bifurcation diagram of $f_\alpha(x)$, in a region of λ corresponding to an odd period ($k=3$) window of $F(x)$. (a) corresponds to $\epsilon=0.001$ for which $\lambda_2^{(k)} < \lambda_{c1}^{(k)}$. (b) corresponds to $\epsilon=0.002$ for which $\lambda_2^{(k)} = \lambda_{c1}^{(k)}$. (c) corresponds to $\epsilon=0.004$ for which $\lambda_2^{(k)} > \lambda_{\max}^{(k)}$. In this case, a second distinct, period-3 window can be seen starting from $\lambda_2^{(k)}$.

basins of attraction, keeps shrinking with increasing ϵ and for $\epsilon \gtrsim 0.1679$ it becomes zero, i.e., $\lambda_2 = \lambda_{c1}$. Thus for $\epsilon \gtrsim 0.1679$, at $\lambda = \lambda_2$, the stable fixed point x_d^* is created within the trajectory of attractor I and the system converges to attractor II, which at birth is in a state of period 1. This process results in the appearance of a discontinuity in the bifurcation diagram. These features are evident in Fig. 2(b), which is the composite bifurcation diagram of $f_\alpha(x)$ for $\epsilon = 0.02$.

Normally for $\lambda_{c1} \le \lambda \le \lambda_{c2}$ all points within the iterative domain of the system converge to attractor II. This is true except for those values of λ in this range for which attractor I has periodic windows. In such cases, for some initial values x_0 the system converges to attractor I and remains there, whereas for other initial x_0 it converges to attractor II. Thus for $\lambda_{c1} \le \lambda \le \lambda_{c2}$, the system will again possess two attractors and show bistability for those subranges of λ for which attractor I possesses periodic windows. For example, the rectangular box in Fig. 2(b) under enlargement can be seen to correspond to a period-2 window of attractor I.

For small ϵ, $\lambda_{max} > \lambda_2$. However on increasing ϵ, we can have $\lambda_{max} < \lambda_2$. For such values of ϵ, the system is not iteratively stable for λ lying in the interval $(\lambda_{max}, \lambda_2)$. A composite bifurcation diagram for $\epsilon \gtrsim 0.2431$ would therefore show an empty region between λ_{max} and λ_2 with the system again becoming iteratively stable and converging to attractor II for $\lambda > \lambda_2$.

V. PERIODIC WINDOWS

The behavior of the map $f_\alpha(x)$ in intervals of λ in which the quadratic map $F(x)$ has windows of period k, depends on whether k is odd or even. When k is odd, at $\lambda_1^{(k)}$, near where the window of $F(x)$ is born, a window of period k is also born to the attractor [which we shall refer to as k(I)] of $f_\alpha(x)$. At a somewhat larger value $\lambda_2^{(k)}$, a second attractor k(II) is born which is also of period k. These attractors are not complementary and with increasing λ undergo the period-doubling route to chaos. Let us denote by $\lambda_{c1}^{(k)}$ and $\lambda_{c2}^{(k)}$ the values of λ at which attractors k(I) and k(II), respectively, collide with the unstable fixed point created with attractor k(II) at $\lambda = \lambda_2^{(k)}$; and by $\lambda_{max}^{(k)}$ the value of λ at which either attractor k(I), or the attractor which exists beyond $\lambda_{c2}^{(k)}$, collides with the unstable fixed point created with attractor k(I) at $\lambda = \lambda_1^{(k)}$. Then, depending on the value of ϵ, three different situations can arise. We illustrate these in Fig. 3, which shows composite bifurcation diagrams of $f_\alpha(x)$ for values of λ in the region of a period-3 window of the map $F(x)$. Figure 3(a) is for small values of ϵ ($\epsilon = 0.001$) when $\lambda_2^{(k)} < \lambda_{c1}^{(k)}$, i.e., attractor k(II) is created before attractor k(I) has undergone its crisis. Figure 3(b) is for somewhat larger values of ϵ ($\epsilon = 0.002$) when $\lambda_2^{(k)} = \lambda_{c1}^{(k)}$ and the system converges to attractor k(II) alone for $\lambda > \lambda_2^{(k)}$. Notice the existence of a period-6 window of the system in the boxed area in the interval $(\lambda_2^{(k)}, \lambda_{c2}^{(k)})$. Figure 3(c) is for even larger values of ϵ ($\epsilon = 0.004$) when $\lambda_2^{(k)} > \lambda_{max}^{(k)}$ and now we see a second distinct period-3 window of the system for $\lambda_2^{(k)} < \lambda < \lambda_{c2}^{(k)}$, which also shows a period-5 window of the system. Box-construction techniques can be used to understand each of these situations.

In regions of λ corresponding to windows of period k (even) of the map $F(x)$, two attractors of the map $f_\alpha(x)$ of period $k/2$, rather than k, get created at $\lambda_1^{(k/2)}$ and $\lambda_2^{(k/2)}$. The other difference from when k is odd is that attractor k(I) never gets to collide with the unstable fixed point that was created together with attractor k(II) because the two are too separated. If we denote by $\lambda_{max}^{(k/2)}$ the value of λ at which attractor k(I) collides with the unstable fixed point that was created with it, then for a given ϵ, two situations can arise, depending on whether $\lambda_{max}^{(k/2)}$ is greater than or less than $\lambda_2^{(k/2)}$. We illustrate them by the composite bifurcation diagrams of $f_\alpha(x)$ in Fig. 4 for values of λ in the region of a period-4 window

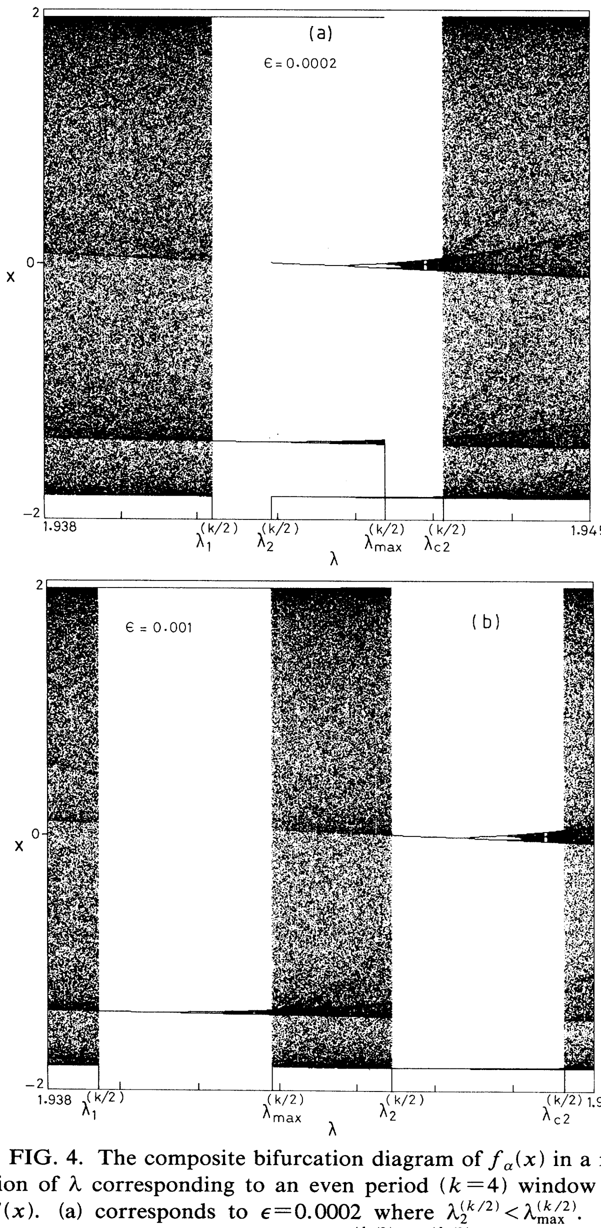

FIG. 4. The composite bifurcation diagram of $f_\alpha(x)$ in a region of λ corresponding to an even period ($k=4$) window of $F(x)$. (a) corresponds to $\epsilon = 0.0002$ where $\lambda_2^{(k/2)} < \lambda_{max}^{(k/2)}$. (b) corresponds to $\epsilon = 0.001$ for which $\lambda_2^{(k/2)} > \lambda_{max}^{(k/2)}$. Note now the occurrence of two distinct period-2 windows.

of the map $F(x)$. In Fig. 4(a) $\epsilon=0.0002$, for which $\lambda_{\max}^{(k/2)} > \lambda_2^{(k/2)}$, and we see first a period-2 window opening up at $\lambda_1^{(k/2)}$ with a second attractor also of basic period 2 being created at $\lambda_2^{(k/2)}$. Figure 4(b) is for a larger value of ϵ ($\epsilon=0.0001$), for which $\lambda_{\max}^{(k/2)} < \lambda_2^{(k/2)}$ and we now see two distinct period-2 windows of the system separated by a region of well developed chaos.

It is clear from the above that the two complementary attractors of $F^{(2)}(x)$ present in a periodic window do not remain complementary in the presence of an additive period-2 forcing term; and the separation between the values of λ at which these attractors are created increases with increasing ϵ. For large enough ϵ, this increasing separation results in the appearance of two separated windows with the same basic period and it may turn out that for sufficiently large ϵ another window of a different basic period may in fact intervene between them. Thus the Sarkovskii ordering [20] of the windows of the modulated map is very different from that of the quadratic map.

VI. ADDITIVE PERIOD-p FORCING

On the basis of our experience with an additive period-2 forcing term we can try and make some generalizations about the quadratic map modulated by an additive forcing term of period p. Although now there are p different period-p maps $f_{p,j}$, $j=0,1,2,\ldots,p-1$ [see Eq. (1.4)], they can be shown to be topological conjugates of each other. Therefore it is sufficient to study any one of them to arrive at the properties of additive period-p forcing.

Since $F^{(p)}(x) \equiv F\circ F\circ F\circ \cdots \circ F$ (p times), $F(x)$ is the $1/p$ cycle map of $F^{(p)}(x)$. For values of λ for which the $1/p$ cycle map is in a state of period n, the map $F^{(p)}(x)$ will possess k complementary attractors, each of period n/k, where k is the highest common factor of p and n. An additive forcing term of period p modulating the quadratic map causes these k complementary attractors of $F^{(p)}(x)$ to become noncomplementary. These attractors can be in states of different periodicities for a given λ since each is born via a tangent bifurcation and undergoes subsequent pitchfork bifurcations at different values of λ. Therefore the system can display multistability, stabilizing to one of a number of possible states ($\leq k$) depending on the initial x_0. The boundaries of the trajectories of these attractors also collide with the unstable fixed points at different values of λ so that the iterative domain, as well as the ranges of λ for which the system is iteratively stable, will now depend in complicated ways on λ and ϵ.

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